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Abstract. Over the years, many researchers have introduced different types of topological spaces. One of these spaces is the octahedron topological spaces, which was introduced and studied by Lee et al. [37]. In this study, very important results were found and it was decided that very important scientific findings could be obtained by developing them. For this reason, some findings such as octahedron topological group in the sense of Forster and relative octahedron homeomorphism which are new scientific data, have been found and proven for the first time in this article. Moreover, the authors are interested to find octahedron quotient space and octahedron quotient topological group. This study is therefore organized the analogies between the concepts of octahedron topological groups, on the other, are strongly emphasized.

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Keywords: Octahedron set, Octahedron topological space, Octahedron subspace, Octahedron continuity, Octahedron quotient space, Octahedron homeomorphism, Octahedron group, Octahedron topological group, Octahedron quotient topological group.

Corresponding Author: S. H. Han, J. G. Lee ({shhan235, jukolee}@wku.ac.kr)

1. Introduction

Historically topological groups arose in connection with the study of groups of continuous transformations. The theory of topological groups is one of the branches of analysis and its study was initiated by Schreier [1] in 1926. In fact, a topological group is obtained by uniting the concepts of group and of topological space. Then the basic facts and concepts pertinent to groups, and to topological spaces are simply translated more or less immediately into the context of topological groups.

In 1965, Zadeh [2] had proposed the concept of fuzzy sets as the generalization of crisp sets in order to deal with various real-life problems. After then, Foster [3] studied initially a fuzzy topological group which unites a fuzzy group in a group with a relative fuzzy topology on the fuzzy group. However, Yu and Ma [4] defined a fuzzy topological group as the ordered pair of a group and a fuzzy topology on the group (See [5–18] for the further studies). In particular, Hošková-Mayerovková [19] proposed the concept of a fuzzy topological hypergroupoid and dealt with some of its properties [20, 21]. Moreover, some researchers [22–25] discussed with fuzzy topological algebraic structures based on some algebras.

In 1983, Atanassov [26] introduced the notion of intuitionistic fuzzy sets as the generalization of fuzzy sets. Hur et al. [27] defined an intuitionistic fuzzy topological group in Foster's sense and obtained some of its basic properties. But Padmapriya et al. [28] investigated topological group structures based on intuitionistic fuzzy sets in the sense of Yu and Ma [4] (See [29–32] for further researches). Abbas [33] introduce the concept of an intuitionistic fuzzy ideal topological group and deal with its various properties.

Recently, in order to reduce possible information loss, Lee et al. [34] proposed an octahedron set combined with an interval-valued fuzzy set, intuitionistic fuzzy set and fuzzy set as a tool to solve complex problems. After that time, Şenel et al. [35] discussed with MCGDM problems by using similarity measures for octahedron sets. Also, Lee et al. [36] proposed the concept of octahedron subgroups and subrings, and obtained some of their properties. Moreover, Lee et al. [37] studied topological structures based on octahedron sets.

Our research's aim is to study topological group structures based on octahedron sets in the sense of Foster [3]. To accomplish this, we proceed with the research in the following order: In Section 2, we recall basic concepts and notations related to octahedron sets. In Section 3, we define an octahedron topology on a set in the sense of Lowen [38] and study further results except properties discussed with Lee et al. [37]. In Section 4, we deal with some results except properties obtained by Lee et al. [36]. Section 5 is devoted to investigate octahedron topological groups.

2. Preliminaries

In this section, we list some basic definitions and notations needed in the next sections. Throughout this paper, I denotes the unit closed interval $[0, 1]$ in the set of real numbers $\mathbb{R}$.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$. Then each member $\bar{a}$ of $I \oplus I$ is called an *intuitionistic point* or *intuitionistic number*. In particuar, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. Refer to [39] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an intuitionistic number, and the infimum and the supremum of any intuitionistic numbers.

Let $[I]$ be the set of all closed subintervals of I. Then each member $\widetilde{a}$ of $[I]$ is called an *interval numbers*, where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$. Refer to [40] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an interval-valued number, and the infimum and the supremum of any interval-valued numbers.

Each member $\widetilde{\bar{a}}$ of $[I] \times (I \oplus I) \times I$ is called an *octahedron numbers*, where

$$\widetilde{\bar{a}} =< \widetilde{a}, \bar{a}, a >=< [a^-, a^-], (a^{\in}, a^{\notin}), a > .$$

The equality and order relation between $\widetilde{\bar{a}}$ and $\widetilde{\bar{b}}$ (See [34]) are defined by:

(i) $(Order)$ $\widetilde{\bar{a}} \leq \widetilde{\bar{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \leq b.$

(i) $(Equality)$ $\widetilde{\bar{a}} = \widetilde{\bar{b}} \Leftrightarrow \widetilde{\bar{a}} \leq \widetilde{\bar{b}}$ and $\widetilde{\bar{b}} \leq \widetilde{\bar{a}}$, i.e., $\widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b},\ a = b.$

For a set X, a mapping $A : X \to I$ is called a *fuzzy set* in X and the set of all fuzzy sets in X is denoted by I^X or $FS(X)$. Refer [2, 41] to basic operations on I^X.

Definition 2.1 ([42]). For a nonempty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IF set) in X, where for each $x \in X$, $\bar{A}(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to $\bar{A}$, respectively. Let $(I \oplus I)^X$ or $IFS(X)$ denote the set of all IF sets in X and for each $\bar{A} \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

Definition 2.2 ([43, 44]). For a nonempty set X, a mapping $\widetilde{A} : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X. Let $[I]^X$ or $IVS(X)$ denote the set of all IVF sets in X. For each $\widetilde{A} \in [I]^X$ and $x \in X$, $\widetilde{A}(x) = [A^-(x), A^+(x)]$ is called the *degree of membership of an element x to $\widetilde{A}$*, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and *an upper fuzzy set* in X, respectively. For each $\widetilde{A} \in [I]^X$, we write $\widetilde{A} = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in X defined by, respectively: for each $x \in X$,

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

Refer to [34, 40] for the definitions of $\subset$ and $=$ on $[I]^X$, the *complement* of an interval-valued set, and the *union* and the *intersection* of any interval-valued sets.

Definition 2.3 ([34]). Let X be a nonempty set and let $\widetilde{A} = [A^-, A^+] \in [I]^X$, $\bar{A} = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$, $A \in I^X$. Then the triple $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle$ is called an *octahedron set* in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping.

We can consider following special octahedron sets in X:

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 0 \right\rangle = \ddot{0}$,

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle,$

$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle,$

$\left\langle \widetilde{1}, \bar{\mathbf{1}}, 1 \right\rangle = \ddot{1}$.

In this case, $\ddot{0}$ (resp. $\ddot{1}$) is called an *octahedron empty set* (resp. *octahedron whole set*) in X. We denote the set of all octahedron sets as $\mathcal{O}(X)$.

Definition 2.4 ([34]). Let X be a nonempty set and let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle$, $\mathcal{B} = \left\langle \widetilde{B}, \bar{B}, B \right\rangle \in \mathcal{O}(X)$. Then the inclusion and the equality between $\mathcal{A}$ and $\mathcal{B}$ are defined by:

(i) (*Inclusion*) $\mathcal{A} \subset \mathcal{B} \Leftrightarrow \widetilde{A} \subset \widetilde{B},\ \bar{A} \subset \bar{B},\ A \subset B$,

(ii) (*Equality*) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$, i.e., $\widetilde{A} = \widetilde{B},\ \bar{A} = \bar{B},\ A = B$.

Definition 2.5 ([34]). Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} = (\left\langle \widetilde{A}_j, \bar{A}_j, A_j \right\rangle)_{j\in J}$ be a family of octahedron sets in X. Then the union $\cup$ and the intersection $\cap$ of $(\mathcal{A}_j)_{j\in J}$ are defined as follows, respectively:

(i) (*Union*) $\bigcup_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{A}_j, \bigcup_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$,

(ii) (*Intersection*) $\bigcap_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{A}_j, \bigcap_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$.

Definition 2.6 ([34]). Let $X,\ Y$ be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} \in \mathcal{O}(X)$, $\mathcal{B} \in \mathcal{O}(Y)$.

(i) The *preimage* of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{B}), f^{-1}(\bar{B}), f^{-1}(B) \right\rangle$, is the octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) = \left\langle [(B^- \circ f)(x), (B^+ \circ f)(x))], ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)), (B \circ f)(x) \right\rangle .$$

(ii) The *image* of $\mathcal{A}$ under f, denoted by $f(\mathcal{A}) = \left\langle f(\widetilde{A}), f(\bar{A}), f(A) \right\rangle$, is the octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\widetilde{A})(y) = \begin{cases} [\bigvee_{x\in f^{-1}(y)} A^-(x), \bigvee_{x\in f^{-1}(y)} A^+(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(\bar{A})(y) = \begin{cases} (\bigvee_{x\in f^{-1}(y)} A^{\in}(x), \bigwedge_{x\in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ \bar{0} & \text{otherwise,} \end{cases}$$

$$f(A)(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $f(x_{\widetilde{\bar{a}}}) = [f(x)]_{\widetilde{\bar{a}}}$, for each $x_{\widetilde{\bar{a}}} \in \mathcal{O}_P(X)$.

Result 2.7 ([34], Proposition 5.5). *Let $\mathcal{A},\ \mathcal{A}_1,\ \mathcal{A}_2 \in \mathcal{O}(X)$, $(\mathcal{A}_j)_{j\in J} \subset \mathcal{O}(X)$, let $\mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2 \in \mathcal{O}(Y)$, $(\mathcal{B}_j)_{j\in J} \subset \mathcal{O}(Y)$ and let $f : X \to Y$ be a mapping. Then*

(1) *if $\mathcal{A}_1 \subset \mathcal{A}_2$, then $f(\mathcal{A}_1) \subset f(\mathcal{A}_2)$,*

(2) *if $\mathcal{B}_1 \subset \mathcal{B}_2$, then $f^{-1}(\mathcal{B}_1) \subset f^{-1}(\mathcal{B}_2)$,*

(3) *$\mathcal{A} \subset f^{-1}(f(\mathcal{A}))$ and if f is injective, then $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,*

(4) *$f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$ and if f is surjective, $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,*

(5) *$f^{-1}(\bigcup_{j\in J} \mathcal{B}_j) = \bigcup_{j\in J} f^{-1}(\mathcal{B}_j)$,*

(6) *$f^{-1}(\bigcap_{j\in J} \mathcal{B}_j) = \bigcap_{j\in J} f^{-1}(\mathcal{B}_j)$,*

(7) *$f(\bigcup_{j\in J} \mathcal{A}_j) = \bigcup_{j\in J} f(\mathcal{A}_j)$,*

(8) *$f(\bigcap_{j\in J} \mathcal{A}_j) \subset \bigcap_{j\in J} f(\mathcal{A}_j)$ and if f is injective, then $f(\bigcap_{j\in J} \mathcal{A}_j) = \bigcap_{j\in J} f(\mathcal{A}_j)$,*

(9) *if f is surjective, then $f(\mathcal{A})^c \subset_1 f(\mathcal{A}^c)$.*

(10) *$f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$.*

(11) *$f^{-1}(\ddot{0}) = \ddot{0}$, $f^{-1}(\ddot{1}) = \ddot{1}$, $f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle$,*

$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle$,
$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle$,
$f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle$.

(12) $f(\ddot{0}) = \ddot{0}$ *and if f is surjective, then the following hold:*

$$f(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \; f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \; f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \; f(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle, \; f(\ddot{1}) = \ddot{1}.$$

3. Further results in octahedron topological spaces

In this section, we define an octahedron topological space in the sense Lowen [38] and obtain some of its properties. In particular, we give a characterization of relative octahedron continuity and a sufficient condition which the relative product mapping is relatively octahedron open.

Definition 3.1 ([37]). Let $\tau \subset \mathcal{O}(X)$. Then τ is called an *octahedron topology* on X, if it satisfies the following axioms:

[OO_1] $\ddot{0}, \; \ddot{1} \in \tau$,
[OO_2] $\mathcal{A} \cap \mathcal{B} \in \tau$ for any $\mathcal{A}, \; \mathcal{B} \in \tau$,
[OO_3] $\bigcup_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$.

It is obvious that the above definition is the sense of Chang [41]. We can give the definition of an octahedron topological space in the sense of Lowen [38] in a natural way. Now we will denote a *constant fuzzy set* [resp. *intuitionistic fuzzy set*, *interval-valued fuzzy set* and *octahedron set*] *with the value* $a \in I$ [resp. $\bar{a} \in I \oplus I$, $\widetilde{a} \in [I]$ and $\widetilde{\bar{a}} \in [I] \times (I \oplus I) \times I$] in a set X as C_a [resp. $C_{\bar{a}}$, $C_{\widetilde{a}}$ and $C_{\widetilde{\bar{a}}}$] and defined by $C_a(x) = a$ [resp. $C_{\bar{a}}(x) = \bar{a}$, $C_{\widetilde{a}}(x)) = \widetilde{a}$ and $C_{\widetilde{\bar{a}}}(x) = \widetilde{\bar{a}}$] for each $x \in X$.

Definition 3.2. Let $\tau \subset \mathcal{O}(X)$. Then τ is called an *octahedron topology* on X in the sense of Lowen, if it satisfies the following axioms:

[LOO_1] $C_{\widetilde{\bar{a}}} \in \tau$,
[LOO_2] $\mathcal{A} \cap \mathcal{B} \in \tau$ for any $\mathcal{A}, \; \mathcal{B} \in \tau$,
[LOO_3] $\bigcup_{j \in J} \mathcal{A}_j \in \tau$ for any $(\mathcal{A}_j)_{j \in J} \subset \tau$.

The pair (X, τ) is called an *octahedron topological space*. The members of τ are called an *octahedron open set* (briefly, an OOS) in X and we denote the set of all OOSs in X as $OOS(X)$. $\mathcal{A} \in \mathcal{O}(X)$ is called an *octahedron closed set* in X, if $\mathcal{A}^c \in \tau$. We denote the collection of all octahedron topologies on X in the sense of Lowen as $OT_L(X)$.

Remark 3.3. Let $FT_L(X)$ [resp. $IFT_L(X)$ and $IVT_L(X)$] denote the set of all fuzzy [resp. intuitionistic fuzzy and interval-valued fuzzy] topologies on X in the sense of Lowen [38] [resp. Çoker [45], and Mondal and Samanta [40]].

Suppose $\tau \in OT_L(X)$. Then we have

$$\tau_{IV} \in IVT_L(X), \; \tau_{IF} \in IFT_L(X), \; \tau_F \in FT_L(X),$$

where $\tau_{IV} = \{\widetilde{U} \in IVS(X) : \mathcal{U} \in \tau\}$, $\tau_{IF} = \{\bar{U} \in IFS(X) : \mathcal{U} \in \tau\}$,
$\tau_{F} = \{U \in FS(X) : \mathcal{U} \in \tau\}$.

Furthermore, we have five fuzzy topologies on X in the sense of Lowen:

$$\tau^{-}_{IV},\ \tau^{+}_{IV},\ \tau^{\in}_{IF},\ \tau^{\notin}_{IF},\ \tau_{F},$$

where $\tau^{-}_{IV} = \{U^{-} \in I^X : \mathcal{U} \in \tau\}$, $\tau^{+}_{IV} = \{U^{+} \in I^X : \mathcal{U} \in \tau\}$,
$\tau^{\in}_{IF} = \{U^{\in} \in I^X : \mathcal{U} \in \tau\}$, $\tau^{\notin}_{IF} = \{U^{\notin c} \in I^X : \mathcal{U} \in \tau\}$ (See Remark 3.3 (1) in [37]).

Example 3.4. (1) Let (X, T) an ordinary topological space and let χ_A denote the characteristic function of a subset A of X. Consider three families given by:

$$\tau_{F,T} = \{\chi_U \in I^X : U \in T\} \cup \{C_a \in I^X : a \in I\},$$

$$\tau_{IF,T} = \{(\chi_U, \chi_{U^c}) \in IFS(X) : U \in T\} \cup \{C_{\bar{a}} \in IFSX : \bar{a} \in I \oplus I\},$$

$$\tau_{IV,T} = \{[\chi_U, \chi_U] \in IVS(X) : U \in T\} \cup \{C_{\widetilde{a}} \in IVSX : \widetilde{a} \in [I]\}.$$

Then we can easily check that $\tau_{F,T} \in FT_L(X)$, $\tau_{IF,T} \in IFT_L(X)$ and $\tau_{IV,T} \in IVT_L(X)$. Furthermore, it is clear that

$$\tau_{O,T} = \{\langle[\chi_U, \chi_U], (\chi_U, \chi_{U^c}), \chi_U\rangle \in \mathcal{O}(X) : U \in T\} \cup \{C_{\widetilde{\bar{a}}} \in \mathcal{O}(X) : \widetilde{\bar{a}} \in [I] \times (I \oplus I) \times I\}$$

is an octahedron topology in Lowens's type.

(2) Let $T \in FT_L(X)$. Consider the following family of octahedron sets in X:

$$\tau_{FT} = \{\langle[U, U], (U, U^c), U\rangle \in \mathcal{O}(X) : U \in T\}.$$

Then clearly, $\tau_{FT} \in OT_L(X)$. In this case, τ_{FT} is called an *octahedron topology on X generated by the fuzzy topology T*.

(3) Let $T \in IVT_L(X)$ and let τ_{IVT} be the family of octahedron sets in X given by:

$$\tau_{IVT} = \left\{\left\langle \widetilde{U}, (U^{-}, U^{+c}), U^{-} \right\rangle \in \mathcal{O}(X) : \widetilde{U} \in T\right\}.$$

Then it is obvious that $\tau_{IVT} \in OT_L(X)$. In this case, τ_{FT} is called an *octahedron topology on X generated by the interval valued fuzzy topology T*.

(4) Let $T \in IFT_L(X)$. Consider the following family of octahedron sets in X:

$$\tau_{IFT} = \left\{\left\langle [U^{\in}, U^{\notin c}], \bar{U}, U^{\in} \right\rangle \in \mathcal{O}(X) : \bar{U} \in T\right\}.$$

Then it is clear that $\tau_{IFT} \in OT_L(X)$. In this case, τ_{FT} is called an *octahedron topology on X generated by the intuitionistic fuzzy topology T*.

(5) Let τ be the set of all constant octahedron set in a set X. Then it is obvious that τ is an octahedron topology. In this case, τ is called a *constant octahedron topology on X* and denoted by τ_{OT}.

Unless otherwise stated, an octahedron [resp. a fuzzy, an intuitionistic fuzzy and interval-valued fuzzy] topological space mean the octahedron topological space in the viewpoint of Lowen in this section and the rest.

Definition 3.5 (See [37])**.** Let (X, τ) and (Y, γ) be two octahedron topological spaces and let $f : X \to Y$ be a mapping. Then f is said to be:

(i) *octahedron continuous*, if $f^{-1}(\mathcal{V}) \in \tau$ for each $\mathcal{V} \in \gamma$,

(ii) *octahedron open*, if $f(\mathcal{A}) \in \gamma$ for each $\mathcal{A} \in \tau$.

Proposition 3.6. *Let (X,τ), (Y,γ) be octahedron topological spaces and let $c : (X,\tau) \to (Y,\gamma)$ the constant mapping $c(x) = y_0 \in Y$ for each $x \in X$. Then c is octahedron continuous.*

Proof. Let $\mathcal{V} \in \gamma$ and let $x \in X$. Then by Definition 2.6 (i),

$$c^{-1}(\mathcal{V})(x) = \mathcal{V}(c(x)) = \mathcal{V}(y_0) = C_{\widetilde{\bar{a}}}(x) \text{ (Say)}.$$

Thus $c^{-1}(\mathcal{V}) = C_{\widetilde{\bar{a}}}$. Since $C_{\widetilde{\bar{a}}} \in \tau$, $c^{-1}(\mathcal{V}) \in \tau$. So c is octahedron continuous. □

The following is an immediate result of Definition 3.5.

Proposition 3.7. *Let (X,τ), (Y,γ), (Z,η) be octahedron topological spaces and let $f : (X,\tau) \to (Y,\gamma)$, $g : (Y,\gamma) \to (Z,\eta)$ be octahedron open. Then $g \circ f : (X,\tau) \to (Z,\eta)$ is octahedron open.*

Definition 3.8 (See Proposition 4.1, [37])**.** Let (X,τ) be an octahedron topological space and let $\mathcal{A} \in \mathcal{O}(X)$. Then the family $\tau_{\mathcal{A}}$ of octahedron sets in X given by:

$$\tau_{\mathcal{A}} = \{\mathcal{A} \cap \mathcal{U} : \mathcal{U} \in \tau\}$$

is called a *relative octahedron topology on $\mathcal{A}$ determined by τ* and the pair $(\mathcal{A}, \tau_{\mathcal{A}})$ is called an *octahedron subspace* of (X,τ). The members of $\tau_{\mathcal{A}}$ is called *relatively octahedron open sets* or simply *octahedron open sets in $\mathcal{A}$*.

Note that the relative octahedron topology does not hold satisfy the axiom [LOO_1] in general. However we can see that it satisfies the axioms [LOO_2] and [LOO_3] from Proposition 4.1 in [37].

Remark 3.9. (1) Let (X,τ) be a fuzzy [resp. an intuitionstic fuzzy and an interval-valued fuzzy] topological space and let $A \in FS(X)$ [resp. $\bar{A} \in IFS(X)$ and $\widetilde{A} \in IVS(X)$. We can define a *relative fuzzy topology* τ_A (See [3]) [resp. *relative intuitionistic fuzzy topology* $\tau_{\bar{A}}$ (See [27]) and *relative interval-valued fuzzy topology* $\tau_{\widetilde{A}}$] on A [resp. $\bar{A}$ and $\widetilde{A}$] as follows:

$$\tau_A = \{U \cap A : U \in \tau\} \text{ [resp. } \tau_{\bar{A}} = \{\{\bar{U} \cap \bar{A} : \bar{U} \in \tau\} \text{ and } \tau_{\widetilde{A}} = \{\{\widetilde{U} \cap \widetilde{A} : \widetilde{U} \in \tau\}].$$

It is obvious that the first axiom of each topology is not satisfied in general.

(2) Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be an octahedron subspace of an octahedron topological space (X,τ). From Remark 3.3 and (1), we can easily check that $(\widetilde{A}, \tau_{IV_{\widetilde{A}}})$ [resp. $(\bar{A}, \tau_{IF_{\bar{A}}})$ and (A, τ_{F_A})] is an interval-valued fuzzy [resp. intuitionistic fuzzy and fuzzy] subspace of an interval-valued fuzzy [resp. an intuitionistic fuzzy and a fuzzy] topological space (X, τ_{IV}) [resp. (X, τ_{IF}) and (X, τ_F)].

Definition 3.10 ([3, 27])**.** (i) Let (X,τ) and (Y,γ) be two fuzzy spaces, let $f : X \to Y$ be a mapping and let (A, τ_A), (B, γ_B) be fuzzy subspaces of (X,τ), (Y,γ) respectively. Then f is called a *mapping of* (A, τ_A) *into* (B, γ_B), denoted by $f : (A, \tau_A) \to (B, \gamma_B)$, if $f(A) \subset B$.

(ii) Let (X,τ) and (Y,γ) be two intuitionistic spaces, let $f : X \to Y$ be a mapping and let $(\bar{A}, \tau_{\bar{A}})$, $(\bar{B}, \gamma_{\bar{A}})$ be intuitionistic fuzzy subspaces of (X,τ), (Y,γ) respectively. Then f is called a *mapping of* $(\bar{A}, \tau_{\bar{A}})$ *into* $(B, \gamma_{\bar{B}})$, denoted by $f : (A, \tau_A) \to (\bar{B}, \tau_{\bar{B}})$, if $f(\bar{A}) \subset \bar{B}$.

We obtain the following modifications corresponding to Definition 3.10.

Definition 3.11. Let (X, τ) and (Y, γ) be two octahedron [resp. interval-valued fuzzy] topological spaces, let $f : X \to Y$ be a mapping, and let $(\mathcal{A}, \tau_{\mathcal{A}})$, $(\mathcal{B}, \gamma_{\mathcal{B}})$ [resp. $(\widetilde{A}, \tau_{\widetilde{A}})$, $(\widetilde{B}, \gamma_{\widetilde{B}})$] be octahedron [resp. interval-valued fuzzy] subspaces of (X, τ), (Y, γ) respectively. Then f is called a *mapping of* $(\mathcal{A}, \tau_{\mathcal{A}})$ [resp. $(\widetilde{A}, \tau_{\widetilde{A}})$] *into* $(\mathcal{B}, \gamma_{\mathcal{B}})$ [resp. $(\widetilde{B}, \gamma_{\widetilde{B}})$], denoted by $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ [resp. $f : (\widetilde{A}, \tau_{\widetilde{A}}) \to (\widetilde{B}, \gamma_{\widetilde{B}})$], if $f(\mathcal{A}) \subset \mathcal{B}$ [resp. $f(\widetilde{A}) \subset \widetilde{B}$].

Definition 3.12 ([3, 27]). (1) Let (X, τ) and (Y, γ) be two fuzzy topological spaces, let (A, τ_A), (B, γ_B) be fuzzy subspaces of (X, τ), (Y, γ) respectively. Then $f : (A, \tau_A) \to (B, \gamma_B)$ is said to be:

(i) *relatively fuzzy continuous*, if $f^{-1}(V) \cap A \in \tau_A$ for each $V \in \gamma_B$,

(ii) *relatively fuzzy open*, if $f(U) \in \gamma_B$ for each $U \in \tau_A$.

(2) Let (X, τ) and (Y, γ) be two intuitionistic fuzzy topological spaces, let $(\bar{A}, \tau_{\bar{A}})$, $(\bar{B}, \tau_{\bar{B}})$ be fuzzy subspaces of (X, τ), (Y, γ) respectively. Then $f : (\bar{A}, \tau_{\bar{A}}) \to (\bar{B}, \tau_{\bar{B}})$ is said to be:

(i) *relatively intuitionistic fuzzy continuous*, if $f^{-1}(\bar{V}) \cap \bar{A} \in \tau_{\bar{A}}$ for each $\bar{V} \in \gamma_{\bar{B}}$,

(ii) *relatively intuitionistic fuzzy open*, if $f(\bar{U}) \in \gamma_{\bar{B}}$ for each $\bar{U} \in \tau_{\bar{A}}$.

Also we obtain the following modifications corresponding to Definition 3.12.

Definition 3.13. Let (X, τ) and (Y, γ) be two octahedron [resp. interval-valued fuzzy] topological spaces, let $(\mathcal{A}, \tau_{\mathcal{A}})$, $(\mathcal{B}, \gamma_{\mathcal{B}})$ [resp. $(\widetilde{A}, \tau_{\widetilde{A}})$, $(\widetilde{B}, \gamma_{\widetilde{B}})$] be octahedron [resp. interval-valued fuzzy] subspaces of (X, τ), (Y, γ) respectively. Then $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ [resp. $f : (\widetilde{A}, \tau_{\widetilde{A}}) \to (\widetilde{B}, \gamma_{\widetilde{B}})$] is said to be:

(i) *relatively octahedron* [resp. *interval-valued fuzzy*] *continuous*, if $f^{-1}(\mathcal{V}) \cap \mathcal{A} \in \tau_{\mathcal{A}}$ [resp. $f^{-1}(\widetilde{V}) \cap \widetilde{A} \in \tau_{\widetilde{A}}$] for each $\mathcal{V} \in \gamma_{\mathcal{B}}$ [resp. $\widetilde{V} \in \gamma_{\widetilde{B}}$],

(ii) *relatively octahedron* [resp. *interval-valued fuzzy*] *open*, if $f(\mathcal{U}) \in \gamma_{\mathcal{B}}$ [resp. $f(\bar{U}) \in \gamma_{\bar{B}}$ and $f(\widetilde{U}) \in \gamma_{\widetilde{B}}$] for each $\mathcal{U} \in \tau_{\mathcal{A}}$ [resp. $\bar{U} \in \tau_{\bar{A}}$ and $\widetilde{U} \in \tau_{\widetilde{A}}$].

From Remark 3.3, and Definitions 3.12 and 3.13, it is obvious that for any two octahedron topological spaces (X, τ) and (Y, γ), if $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \tau_{\mathcal{B}})$ is relatively octahedron continuous [resp. open], then $f : (\widetilde{A}, \tau_{IV_{\widetilde{A}}}) \to (\widetilde{B}, \tau_{IV_{\widetilde{B}}})$ is a relatively interval-valued fuzzy continuous [resp. open], $f : (\bar{A}, \tau_{IF_{\bar{A}}}) \to (\bar{B}, \tau_{IF_{\bar{B}}})$ is a relatively intuitionistic fuzzy continuous [resp. open] and $f : (A, \tau_{F_A}) \to (B, \tau_{F_B})$ is a relatively fuzzy continuous [resp. open].

Proposition 3.14. *Let (X, τ) and (Y, γ) be two octahedron topological spaces, let $(\mathcal{A}, \tau_{\mathcal{A}})$, $(\mathcal{B}, \gamma_{\mathcal{B}})$ be octahedron subspaces of (X, τ), (Y, γ) respectively and let $f : (X, \tau) \to (Y, \gamma)$ be octahedron continuous such that $f(\mathcal{A}) \subset \mathcal{B}$. Then $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ is a relatively octahedron continuous.*

Proof. Let $\mathcal{V} \in \gamma_{\mathcal{B}}$. Then clearly, there is $\mathcal{U} \in \gamma$ such that $\mathcal{V} = \mathcal{U} \cap \mathcal{B}$. Since f is octahedron continuous, $f^{-1}(\mathcal{U}) \in \tau$. On the other hand, by Result 2.7 (6), we have

$$f^{-1}(\mathcal{V}) \cap \mathcal{A} = f^{-1}(\mathcal{U}) \cap f^{-1}(\mathcal{B}) \cap \mathcal{A}.$$

Since $\mathcal{V} = \mathcal{U} \cap \mathcal{B}$, $f^{-1}(\mathcal{U}) \cap f^{-1}(\mathcal{B}) \cap \mathcal{A} = f^{-1}(\mathcal{U}) \cap \mathcal{A}$. Thus $f^{-1}(\mathcal{V}) \cap \mathcal{A} = f^{-1}(\mathcal{U}) \cap \mathcal{A}$. So $f^{-1}(\mathcal{V}) \cap \mathcal{A} \in \tau_{\mathcal{A}}$. Hence the result holds. □

Proposition 3.15. *Let (X, τ), (Y, γ), (Z, η) be octahedron topological spaces, let $(\mathcal{A}, \tau_{\mathcal{A}})$, $(\mathcal{B}, \gamma_{\mathcal{B}})$, $(\mathcal{C}, \eta_{\mathcal{C}})$ be octahedron subspaces of (X, τ), (Y, γ), (Z, η) respectively. Suppose $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ and $g : (\mathcal{B}, \gamma_{\mathcal{B}}) \to (\mathcal{C}, \eta_{\mathcal{C}})$ are relatively octahedron continuous [resp. relatively octahedron open]. Then $g \circ f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{C}, \eta_{\mathcal{C}})$ is relatively octahedron continuous [resp. relatively octahedron open].*

Proof. Suppose f and g are relatively octahedron continuous. Let $W^{'} \in \eta_{\mathcal{C}}$. Then $g^{-1}(W^{'}) \cap \mathcal{B} \in \gamma_{\mathcal{B}}$. Thus we have

$$f^{-1}(g^{-1}(W^{'}) \cap \mathcal{B}) \cap \mathcal{A} \in \tau_{\mathcal{A}}.$$

Since $f(\mathcal{A}) \subset \mathcal{B}$, we get

$$f^{-1}(g^{-1}(W^{'}) \cap \mathcal{B}) \cap \mathcal{A} = (g \circ f)^{-1}(W^{'}) \cap \mathcal{A}.$$

So $g \circ f$ is relatively octahedron continuous. The remainder's proof is obvious. □

Definition 3.16 ([37]). Let τ_1, $\tau_2 \in OT_L(X)$. Then we say that τ_1 is *coarser than* τ_2 or τ_2 is *finer than* τ_1, if $\tau_1 \subset \tau_2$, i.e., the identity mapping $id : (X, \tau_2) \to (X, \tau_1)$ is octahedron continuous.

Definition 3.17 ([37]). Let (X, τ) be an octahedron topological space and let $\beta \subset \tau$. Then β is called an *octahedron base* for τ, if for each $\mathcal{U} \in \tau$, $\mathcal{U} = \ddot{0}$ or there is a $\beta^{'} \subset \beta$ such that $\mathcal{U} = \bigcup \beta^{'}$, i.e., each member of τ can be expressed as the union of β.

Definition 3.18. Let $\tau \in OT_L(X)$, let $\mathcal{A} \in \mathcal{O}(X)$ and let $\beta^{'} \subset \tau_{\mathcal{A}}$ Then $\beta^{'}$ is called an *octahedron base* for $\tau_{\mathcal{A}}$, if each member of $\tau_{\mathcal{A}}$ can be expressed as the union of $\beta^{'}$.

It is well-known that if β is an octahedron topology on a set X and $\mathcal{A} \in \mathcal{O}(X)$, then $\beta_{\mathcal{A}} = \{\mathcal{B} \cap \mathcal{A} : \mathcal{B} \in \beta\}$ is an octahedron base for $\tau_{\mathcal{A}}$ (See Proposition 4.5 in [37]).

From Definitions 3.5, 3.17 and 3.18, we have the following results.

Theorem 3.19 (See Theorem 5.6, [37]). *Let (X, τ), (Y, γ) be octahedron topological spaces, let $f : (X, \tau) \to (Y, \gamma)$ be a mapping and let $\beta \subset \gamma$. Then f is octahedron continuous if and only if $f^{-1}(\mathcal{B}) \in \tau$ for each $\mathcal{B} \in \beta$.*

Theorem 3.20. *Let (X, τ), (Y, γ) be octahedron topological spaces, let $(\mathcal{A}, \tau_{\mathcal{A}})$, $(\mathcal{B}, \gamma_{\mathcal{B}})$ be octahedron subspaces of (X, τ), (Y, γ) respectively and let $\beta^{'} \subset \gamma_{\mathcal{B}}$ Then a mapping $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ is relatively octahedron continuous if and only if $f^{-1}(\mathcal{B}^{'}) \cap \mathcal{A} \in \tau_{\mathcal{A}}$ for each $\mathcal{B}^{'} \in \beta^{'}$.*

Definition 3.21 (See Proposition 5.13, [37]). Let $f : X \to Y$ be a mapping and let $\gamma \in OT_L(Y)$. Then the coarsest octahedron topology τ on X for which $f : (X, \tau) \to (Y, \gamma)$ is octahedron continuous is called the *initial octahedron topology* on X or the *inverse image under f* of γ. In fact, $\tau = f^{-1}(\gamma) = \{f^{-1}(\mathcal{U}) \in \mathcal{O}(X) : \mathcal{U} \in \gamma\}$.

Definition 3.22 (See Proposition 5.7, [37]). Let $f : X \to Y$ be a mapping and let $\tau \in OT_L(X)$. Then the finest octahedron topology γ on Y for which $f : (X, \tau) \to (Y, \gamma)$ is octahedron continuous is called the *final octahedron topology* on X or the *image under f* of τ. In fact, $\gamma = f(\tau) = \{\mathcal{V} \in \mathcal{O}(Y) : f^{-1}(\mathcal{V}) \in \tau\}$.

Definition 3.23 (See [37]). Let $(X_j, \tau_j)_{j\in J}$ be a family of octahedron topological spaces, let $X = \Pi_{j\in J}X_j$ and let $(\pi_j : X \to (X_j, \tau_j))_{j\in J}$ be a family of mappings, where π_j is the projection mapping. Then the initial octahedron topology τ on X induced by $(\pi_j)_{j\in J}$ is called the *octahedron product topology* on X and denoted by $\tau = \Pi_{j\in J}\tau_j$.

Proposition 3.24. *Let $(X_j, \tau_j)_{j\in J}$ be a family of octahedron topological spaces, let (X, τ) be the octahedron product topological space. Then the set of all finite intersections of octahedron sets of the form $\pi_j^{-1}(\mathcal{U}_j)$ is an octahedron base for τ, where $\mathcal{U}_j \in \tau_j$, $j \in J$.*

Proof. The proof is similar to one of such classical case. □

Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a family of sets and let $\mathcal{A}_j \in \mathcal{O}(X_j)$ for each $j = 1, 2, \cdots, n$. Then the *octahedron product* of $\{\mathcal{A}_j\}$, $j = 1, 2, \cdots, n$, denoted by $\Pi^n_{j=1}\mathcal{A}_j$, is an octahedron set in $X = \Pi^n_{j=1}X_j$ defined as follows: for each $(x_1, x_2, \cdots, x_n) \in X$,

$$\mathcal{A}(x_1, x_2, \cdots, x_n) = \mathcal{A}_1(x_1) \wedge \mathcal{A}_2(x_2) \wedge \cdots \wedge \mathcal{A}_n(x_n),$$

where $\mathcal{A} = \Pi^n_{j=1}\mathcal{A}_j$. Then we can easily check that $\pi_j(\mathcal{A}) \subset \mathcal{A}_j$ for each $j = 1, 2, \cdots, n$.

Remark 3.25. From Proposition 3.24, it follows that if $\tau_j \in OT_L(X_j)$, $j = 1, 2, \cdots, n$, then the set of octahedron product sets of the form $\Pi^n_{j=1}\mathcal{U}_j$ is an octahedron base for the octahedron product topology on X, where $\mathcal{U}_j \in \tau_j$, $j = 1, 2, \cdots, n$.

Proposition 3.26. *Let $(X_j, \tau_j)_{j\in J}$ be a family of octahedron topological spaces and let (X, τ) be the octahedron product topological space. For each $j = 1, 2, \cdots, n$, let $\mathcal{A}_1 \in \mathcal{O}(X_j)$, let $\mathcal{A}$ be the octahedron product set in X and let $\tau_{\mathcal{A}}$ be the relative octahedron topology on $\mathcal{A}$. Then the set of octahedron product sets of the form $\Pi^n_{j=1}\mathcal{U}'_j$ is an octahedron base for $\tau_{\mathcal{A}}$, where $\mathcal{U}'_j \in \tau_{j_{\mathcal{A}}}$, $j = 1, 2, \cdots, n$.*

Proof. From Remark 3.25, it is clear that the following family of octahedron sets in X

$$\beta = \{\Pi^n_{j=1}\mathcal{U}_j \in \mathcal{O}(X) : \mathcal{U}_j \in \tau_j,\ j = 1, 2, \cdots, n\}$$

is an octahedron base for τ. Then we can easily see that

$$\beta_{\mathcal{A}} = \{(\Pi^n_{j=1}\mathcal{U}_j) \cap \mathcal{A} : \mathcal{U}_j \in \tau_j,\ j = 1, 2, \cdots, n\}$$

is an octahedron base for $\tau_{\mathcal{A}}$. On the other hand, $(\Pi^n_{j=1}\mathcal{U}_j) \cap \mathcal{A} = \Pi^n_{j=1}(\mathcal{U}_j \cap \mathcal{A})$. Thus the result follows that $\mathcal{U}'_j = \mathcal{U}_j \cap \mathcal{A} \in \tau_{j_{\mathcal{A}}}$, $j = 1, 2, \cdots, n$. □

Theorem 3.27 (See Theorem 5.19, [37]). *Let $(X_j, \tau_j)_{j\in J}$ be a family of octahedron topological spaces, let (X, τ) be the octahedron product topological space and let (Y, γ) be an octahedron topological space. Then a mapping $f : (Y, \gamma) \to (X, \tau)$ is octahedron continuous if and only if $\pi_j \circ f$ is octahedron continuous for each $j \in J$.*

Proof. The proof is almost similar to one of Theorem 5.19 in [37]. □

Corollary 3.28. *Let $(X_j, \tau_j)_{j\in J}$, $(Y_j, \gamma_j)_{j\in J}$ be two families of octahedron topological spaces, and let (X, τ), (Y, γ) be the respective octahedron product topological space. For each $j \in J$, let $f_j : (X_j, \tau_j) \to (Y_j, \gamma_j)$ be a mapping and let $f = \Pi_{j\in J} f_j$: $(X, \tau) \to (Y, \gamma)$ be the product mapping defined by: for each $x = (x_j) \in X$,*

$$f(x) = (f_j(x_j)).$$

If f_j is octahedron continuous for each $j \in J$, then f is octahedron continuous.

Proof. Let $x = (x_j) \in X$. Then clearly, $f(x) = (f_j(\pi_j(x))) = (f_j \circ \pi_j)(x)$, i.e., $f = f_j \circ \pi_j$. Thus by Theorem 3.27, $f_j \circ \pi_j$ is octahedron continuous for each $j \in J$. So f is octahedron continuous. □

Theorem 3.29. *Let $(X_j, \tau_j)_{j\in J}$ be a finite family of octahedron topological spaces, $j = 1, 2, \cdots, n$, let (X, τ) be the octahedron product topological space and let (Y, γ) be an octahedron topological space. Let $f : (\mathcal{B}, \gamma_{\mathcal{B}}) \to (\mathcal{A}, \tau_{\mathcal{A}})$ be a mapping. Then f is relatively octahedron continuous if and only if $\pi_j \circ f$: is relatively octahedron continuous for each $j = 1, 2, \cdots, n$.*

Proof. Suppose f is relatively octahedron continuous. It is clear that $\pi_j : (X, \tau) \to (X_j, \tau_j)$ is octahedron continuous such that $\pi_j(\mathcal{A}) \subset \mathcal{A}_j$ for each $j \in J$. Then by Proposition 3.14, $\pi_j : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{A}_j, (\tau_j)_{\mathcal{A}_j})$ is relatively octahedron continuous for each $j \in J$. Thus $\pi_j \circ f$: is relatively octahedron continuous for each $j \in J$.

Conversely, suppose $\pi_j \circ f$: is relatively octahedron continuous for each $j \in J$ and let $\mathcal{U}' = \Pi_{j=1}^n \mathcal{U}_j'$, where $\mathcal{U}_j' \in \tau_{j_{\mathcal{A}_j}}$, $j = 1, 2, \cdots, n$. Then by Proposition 3.26, the set of such $\mathcal{U}'$ forms an octahedron base for $\tau_{\mathcal{A}}$. On the other hand,

$$\begin{aligned} f^{-1}(\mathcal{U}') \cap \mathcal{B} &= f^{-1}(\pi_1^{-1}(\mathcal{U}_1') \cap \pi_2^{-1}(\mathcal{U}_2') \cap \cdots \cap (\pi_n^{-1}(\mathcal{U}_n')) \cap \mathcal{B} \\ &= \bigcap_{j=1}^n ((\pi_j \circ f)^{-1}(\mathcal{U}_j') \cap \mathcal{B}). \end{aligned}$$

Thus by the hypothesis, $(\pi_j \circ f)^{-1}(\mathcal{U}_j') \cap \mathcal{B} \in \gamma_{\mathcal{B}}$ for each $j \in J$. So $f^{-1}(\mathcal{U}') \cap \mathcal{B} \in \gamma_{\mathcal{B}}$. So by Theorem 3.20, f is relatively octahedron continuous. □

Corollary 3.30. *Let $(X_j, \tau_j)_{j\in J}$, $(Y_j, \gamma_j)_{j\in J}$ be two families of octahedron topological spaces, and let (X, τ), (Y, γ) be the respective octahedron product topological space. For each $j \in J$, let $\mathcal{A}_j \in \mathcal{O}(X_j)$, $\mathcal{B}_j \in \mathcal{O}(Y_j)$ and let $f_j : (\mathcal{A}_j, (\tau_j)_{\mathcal{A}_j}) \to (\mathcal{B}_j, (\gamma_j)_{\mathcal{B}_j})$ be a mapping. Let $\mathcal{A} = \Pi_{j=1}^n \mathcal{A}_j$, $\mathcal{B} = \Pi_{j=1}^n \mathcal{B}_j$ be the octahedron product sets in X, Y respectively and let $f = \Pi_{j\in J} f_j : (X, \tau) \to (Y, \gamma)$ be the product mapping defined in Corollary 3.28. If f_j is relatively octahedron continuous for each $j = 1, 2, \cdots, n$, then $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ is relatively octahedron continuous.*

Proof. The proof is analogous to one of Corollary 3.28. □

Proposition 3.31. *Let $(X_j, \tau_j)_{j\in J}$, $(Y_j, \gamma_j)_{j\in J}$ be two families of octahedron topological spaces and let (X, τ), (Y, γ) be the respective octahedron product topological space. For each $j \in J$, let $f_j : (X_j, \tau_j) \to (Y_j, \gamma_j)$ be a mapping. Let $f = \Pi_{j\in J} f_j$: $(X, \tau) \to (Y, \gamma)$ be the product mapping defined in Corollary 3.28. If f_j is octahedron open for each $j = 1, 2, \cdots, n$, then f is octahedron open.*

Proof. Suppose f_j is octahedron open for each $j \in J$ and let $\mathcal{U} \in \tau$. Then there are $\mathcal{U}_{j_m} \in \tau_j$, $m \in M$, $j = 1, 2, \cdots, n$ such that

$$\mathcal{U} = \bigcup_{m \in M} \Pi_{j=1}^{n} \mathcal{U}_{j_m}.$$

Since $\mathcal{U} = \left\langle \widetilde{U}, \bar{U}, U \right\rangle$ and $\mathcal{U}_{j_m} = \left\langle \widetilde{U}_{j_m}, \bar{U}_{j_m}, U_{j_m} \right\rangle$, we have

$$\widetilde{U} = \bigcup_{m \in M} \Pi_{j=1}^{n} \widetilde{U}_{j_m}, \ \bar{U} = \bigcup_{m \in M} \Pi_{j=1}^{n} \bar{U}_{j_m}, \ U = \bigcup_{m \in M} \Pi_{j=1}^{n} U_{j_m}.$$

Thus $f(\mathcal{U}) = \left\langle f(\widetilde{U}), f(\bar{U}), f(U) \right\rangle$. Let $y \in Y$ such that $f^{-1}(y) \neq \varnothing$. Then by Definition 2.6 (ii), we have the following:

$$\begin{aligned} f(\widetilde{U})(y) &= f(\textstyle\bigcup_{m \in M} \Pi_{j=1}^{n} \widetilde{U}_{j_m})(y) \\ &= \textstyle\bigvee_{m \in M} [\bigvee_{x \in f^{-1}(y)} \Pi_{j=1}^{n} U_{j_m}^{-}(x), \bigvee_{x \in f^{-1}(y)} \Pi_{j=1}^{n} U_{j_m}^{+}(x)] \\ &= \textstyle\bigvee_{m \in M} [\bigvee_{x_1 \in f^{-1}(y_1)} \cdots \bigvee_{x_n \in f^{-1}(y_n)} (U_{1_m}^{-}(x_1) \wedge \cdots \wedge U_{n_m}^{-}(x_n)), \\ &\qquad \textstyle\bigvee_{x_1 \in f^{-1}(y_1)} \cdots \bigvee_{x_n \in f^{-1}(y_n)} (U_{1_m}^{+}(x_1) \wedge \cdots \wedge U_{n_m}^{+}(x_n))] \\ &= \textstyle\bigvee_{m \in M} [(\bigvee_{x_1 \in f^{-1}(y_1)} U_{1_m}^{-}(x_1)) \wedge \cdots \wedge (\bigvee_{x_n \in f^{-1}(y_n)} U_{n_m}^{-}(x_n)), \\ &\qquad (\textstyle\bigvee_{x_1 \in f^{-1}(y_1)} U_{1_m}^{+}(x_1)) \wedge \cdots \wedge (\bigvee_{x_n \in f^{-1}(y_n)} U_{n_m}^{+}(x_n))] \\ &= \textstyle\bigvee_{m \in M} [f(U_{1_m}^{-})(y_1) \wedge \cdots f(U_{n_m}^{-})(y_n), \\ &\qquad f(U_{1_m}^{+})(y_1) \wedge \cdots f(U_{n_m}^{+})(y_n)] \\ &= \textstyle\bigvee_{m \in M} (\Pi_{j=1}^{n} f_j(\widetilde{U}_{j_m})(y)) \\ &= (\textstyle\bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\widetilde{U}_{j_m}))(y). \end{aligned}$$

Similarly, we get $f(\bar{U})(y) = (\bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\bar{U}_{j_m}))(y)$. Also by the proof of Proposition 3.6 in [3], we obtain $f(U)(y) = (\bigcup_{m \in M} \Pi_{j=1}^{n} f_j(U_{j_m}))(y)$. Thus we have

$$f(\widetilde{U}) = \bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\widetilde{U}_{j_m}), \ f(\bar{U})(y) = \bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\bar{U}_{j_m}),$$

$$f(U) = \bigcup_{m \in M} \Pi_{j=1}^{n} f_j(U_{j_m}).$$

So $f(\mathcal{U}) = \bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\mathcal{U}_{j_m})$. Since $f_j(\mathcal{U}_{j_m})$ is octahedron open for each $j = 1, 2, \cdots, n$, $f(\mathcal{U}) \in \gamma$. Hence f is octahedron open. □

Proposition 3.32. *Let $(X_j, \tau_j)_{j \in J}$, $(Y_j, \gamma_j)_{j \in J}$ be two families of octahedron topological spaces and let (X, τ), (Y, γ) be the respective octahedron product topological space. For each $j \in J$, let $\mathcal{A}_j \in \mathcal{O}(X_j)$, $\mathcal{B}_j \in \mathcal{O}(Y_j)$ and let $f_j : (\mathcal{A}_j, (\tau_j)_{\mathcal{A}_j}) \to (\mathcal{B}_j, (\gamma_j)_{\mathcal{B}_j})$ be a mapping. Let $\mathcal{A} = \Pi_{j=1}^{n} \mathcal{A}_j$, $\mathcal{B} = \Pi_{j=1}^{n} \mathcal{B}_j$ be the octahedron product sets in X, Y respectively and let $f = \Pi_{j \in J} f_j : (X, \tau) \to (Y, \gamma)$ be the product mapping defined in Corollary 3.28. If f_j is relatively octahedron open for each $j = 1, 2, \cdots, n$, then f is relatively octahedron open.*

Proof. Suppose f_j is relatively octahedron open for each $j = 1, 2, \cdots, n$ and let $\mathcal{U}' \in \tau_{\mathcal{A}}$. Then by Proposition 3.26, there are $\mathcal{U}'_{j_m} \in (\tau_j)_{\mathcal{A}_j}$, $m \in M$, $j = 1, 2, \cdots, n$ such that

$$\mathcal{U}' = \bigcup_{m \in M} \mathcal{U}'_{j_m}.$$

As in the proof of Proposition 3.31, we get

$$f(\mathcal{U}^{'}) = \bigcup_{m \in M} \Pi_{j=1}^{n} f_j(\mathcal{U}^{'}_{j_m}).$$

Thus by the hypothesis, $f(\mathcal{U}^{'}) \in \gamma_{\mathcal{B}}$. So f is relatively octahedron open. □

Proposition 3.33. *Let (X_1, τ_1), (X_2, τ_2) be two octahedron topological spaces and let (X, τ) be the octahedron product topological space. For each $a_1 \in X_1$, let $i : (X_2, \tau_2) \to (X, \tau)$ be the mapping given by: for each $x_2 \in X_2$,*

$$i(a_1) = (a_1, x_2).$$

Then i is octahedron continuous.

Proof. Consider the constant mapping $c : (X_2, \tau_2) \to (X_1, \tau_1)$ given by $c(x_2) = a_1$ for each $x_2 \in X_2$. Then by Proposition 3.6, c is octahedron continuous. From by Proposition 5.4 (1) in [37], it is clear that the identity mapping $id : (X_2, \tau_2) \to (X_2, \tau_2)$ is octahedron continuous. Thus by Theorem 3.27, $i = c \circ id$ is octahedron continuous. □

Proposition 3.34. *Let (X_1, τ_1), (X_2, τ_2) be two octahedron topological spaces and let (X, τ) be the octahedron product topological space. Let $\mathcal{A}_1 \in \mathcal{O}(X_1)$, $\mathcal{A}_2 \in \mathcal{O}(X_2)$,let $\mathcal{A}$ be the octahedron product set in X and let $i : (X_2, \tau_2) \to (X, \tau)$ be the mapping given in Proposition 3.33. Suppose $\mathcal{A}_1(a_1) \geq \mathcal{A}_2(x_2)$ for each $a_1 \in X_1$ and each $x_2 \in X_2$. Then $i : (\mathcal{A}_2, {\tau_2}_{\mathcal{A}_2}) \to (\mathcal{A}, \tau_{\mathcal{A}})$ is relatively octahedron continuous.*

Proof. From Definition 2.6 (ii) and the hypothesis, we can easily check that $i(\mathcal{A}_2) \subset \mathcal{A}$. Then $i : (\mathcal{A}_2, {\tau_2}_{\mathcal{A}_2}) \to (\mathcal{A}, \tau_{\mathcal{A}})$. Thus by the proof of the octahedron continuity of i in Proposition 3.33, we can show that i is relatively octahedron continuous. □

4. Further properties of octahedron subgroups

In this section, first of all, we recall some definitions and results with respect to octahedron subgroups introduced by Lee et al. [36]. Next, we find some additional properties for octahedron subgroups.

Definition 4.1 (See [36])**.** Let X be a group and let $\mathcal{G} \in \mathcal{O}(X)$. Then $\mathcal{G}$ is called an *octahedron group* in X, if it satisfies the following axioms:
(i) $\mathcal{G}(xy) \geq \mathcal{G}(x) \wedge \mathcal{G}(y)$ for any $x,\ y \in X$,
(ii) $\mathcal{G}(x^{-1}) \geq \mathcal{G}(x)$ for each $x \in X$.
We denote the set of all octahedron groups in X as $OG(X)$.

Note that Lee et al. [36] refers to G as an *octahedron subgroup* of X.

Result 4.2 (Proposition 17, [36])**.** *Let $\mathcal{G}$ be an octahedron group in a group X. Then $\mathcal{G}(x) = \mathcal{G}(x^{-1})$ and $\mathcal{G}(e) \geq \mathcal{G}(x)$ for each $x \in X$, where e is the identity element of X.*

Result 4.3 (Theorem 7, [36])**.** *Let X be a group and let $\mathcal{G} \in \mathcal{O}(X)$. Then $\mathcal{G} \in OG(X)$ if and only if $\mathcal{G}(xy^{-1}) \geq \mathcal{G}(x) \wedge \mathcal{G}(y)$ for any $x,\ y \in X$.*

Definition 4.4 ([36]). Let X be a non-empty set and let $\mathcal{A} \in \mathcal{O}(X)$. Then $\mathcal{A}$ is said to *have sup-property*, if for each $T \in 2^X$, there is $t_0 \in T$ such that

$$\mathcal{A}(t_0) = \bigvee_{t \in T} \mathcal{A}(t) = \left\langle \bigvee_{t \in T} \widetilde{A}(t), \bigvee_{t \in T} \bar{A}(t), \bigvee_{t \in T} A(t) \right\rangle.$$

It is obvious that $\mathcal{A}$ has sup-property if and only if $\widetilde{A}$, $\bar{A}$ and A have sup-property respectively (See [46–48]).

Result 4.5 (Proposition 12, [36]). *Let $f : X \to Y$ be a group homomorphism and let $\mathcal{G} \in OG(X)$, $\mathcal{G}' \in OG(Y)$.*
(1) *If $\mathcal{G}$ has the sup-property, then $f(\mathcal{G}) \in OG(Y)$.*
(2) *$f^{-1}(\mathcal{G}') \in OG(X)$.*

Definition 4.6 ([36]). Let X, Y be sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} \in \mathcal{O}(X)$. Then $\mathcal{A}$ is said to be *f-invariant*, if for any x, $y \in X$, $f(x) = f(y)$ implies $\mathcal{A}(x) = \mathcal{A}(y)$.

It is clear that $\mathcal{A}$ is f-invariant if and only if $\widetilde{A}$, $\bar{A}$ and A are f-invariant respectively (See [46, 47, 49]). Furthermore, we can easily check that if $\mathcal{A}$ is f-invariant, then $f^{-1}(f(\mathcal{A})) = \mathcal{A}$.

The following is an immediate consequence of Definition 4.6.

Proposition 4.7. *Let $f : X \to Y$ be a group homomorphism and let $\mathcal{G} \in OG(X)$. If $\mathcal{G}$ is f-invariant, then $f(\mathcal{G}) \in OG(Y)$.*

Result 4.8 (Proposition 19, [36]). *Let X be a group and let $\mathcal{G} \in OG(X)$. Then*

$$\mathcal{G}_e = \{x \in X : \mathcal{G}(x) = \mathcal{G}(e)\}$$

is a subgroup of X.

For a fixed $a \in X$, let r_a, $l_a : X \to X$ two mappings defined respectively as follows: for each $x \in X$,

$$r_a(x) = xa, \ l_a(x) = ax.$$

Then r_a and l_a are called the *right* and *left translations of X into itself.*

Proposition 4.9. *Let X be a group and let $\mathcal{G} \in OG(X)$. Then we have*

$$r_a(\mathcal{G}) = l_a(\mathcal{G}) = \mathcal{G} \text{ for each } a \in \mathcal{G}_e.$$

Proof. Let $a \in \mathcal{G}_e$ and let $x \in X$. Then we get

$$\begin{aligned} r_a(\mathcal{G})(x) &= \left\langle \bigvee_{y \in r_a^{-1}} \widetilde{G}(y), \bigvee_{y \in r_a^{-1}} \bar{G}(y), \bigvee_{y \in r_a^{-1}} G(y) \right\rangle \text{ [By Definition 2.6]} \\ &= \left\langle \widetilde{G}(xa^{-1}), \bar{G}(xa^{-1}), G(xa^{-1}) \right\rangle \text{ [Since } r_a^{-1}(x) = xa^{-1}] \\ &= \mathcal{G}(xa^{-1}) \\ &\geq \mathcal{G}(x) \wedge \mathcal{G}(a) \text{ [By Result 4.3]} \\ &= \mathcal{G}(x) \wedge \mathcal{G}(e) \text{ [Since } a \in \mathcal{G}_e] \\ &= \mathcal{G}(x) \text{ [By Result 4.2]} \\ &= \mathcal{G}(xa^{-1}a) \text{ [Since } X \text{ is a group]} \\ &\geq \mathcal{G}(xa^{-1}) \wedge \mathcal{G}(a) \text{ [By Definition 4.11]} \\ &= \mathcal{G}(xa^{-1}) \wedge \mathcal{G}(e) \text{ [Since } a \in \mathcal{G}_e] \end{aligned}$$

$= \mathcal{G}(xa^{-1})$ [By Result 4.2]
$= r_a(\mathcal{G})(x)$.
Thus $r_a(\mathcal{G}) = \mathcal{G}$. The proof for l_a is similar. □

Definition 4.10 (See [36]). Let X be a group and let $\mathcal{A},\ \mathcal{B} \in \mathcal{O}(X)$.

(i) The *product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B}$, is the octahedron set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \circ \mathcal{B})(x) = \bigvee_{x_1 x_2 = x} [\mathcal{A}(x_1) \wedge \mathcal{B}(x_2)].$$

(ii) The inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{-1}$, is the octahedron set in X defined as follows: for each $x \in X$,

$$\mathcal{A}^{-1}(x) = \mathcal{A}(x^{-1}).$$

The following is an immediate consequence of Definition 4.10.

Proposition 4.11. *Let X be a group and let $\mathcal{A},\ \mathcal{B} \in \mathcal{O}(X)$. Then*

$$(\mathcal{A} \circ \mathcal{B})^{-1} = \mathcal{B}^{-1} \circ \mathcal{A}^{-1},\ (\mathcal{A}^{-1})^{-1}.$$

Proposition 4.12. *Let X be a group and let $\mathcal{A},\ \mathcal{B} \in \mathcal{O}(X)$.*
(1) *If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{A} \circ \mathcal{C} \subset \mathcal{B} \circ \mathcal{C}$, $\mathcal{C} \circ \mathcal{A} \subset \mathcal{C} \circ \mathcal{B}$ for each $\mathcal{C} \in \mathcal{O}(X)$.*
(2) *If $\mathcal{A} \circ \mathcal{C} = \mathcal{B} \circ \mathcal{C}$ for each $\mathcal{C} \in \mathcal{O}(X)$, then $\mathcal{A} = \mathcal{B}$.*
(3) *If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{A}^{-1} \subset \mathcal{B}^{-1}$.*
(4) *The following holds: for any $\mathcal{A},\ \mathcal{B},\ \mathcal{B} \in \mathcal{O}(X)$,*

$$(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C}).$$

Proof. The proofs of (1), (3) and (4) are easy.
(2) Assume that $\mathcal{A} \neq \mathcal{B}$. Then there is $x \in X$ such that

$$\widetilde{A}(x) \neq \widetilde{B}(x) \text{ or } \bar{A}(x) \neq \bar{B}(x) \text{ or } A(x) \neq B(x).$$

Let $\mathcal{C}$ be the octahedron set in X given by: for each $y \in X$,

$$\mathcal{C}(y) = \begin{cases} \langle [1,1], (1,0), 1 \rangle & \text{if } y = e \\ \langle [0,0], (0,1), 0 \rangle & \text{if } y \neq e, \end{cases}$$

where e is the identity element of X. Then we can easily see that $\mathcal{A} \circ \mathcal{C} \neq \mathcal{B} \circ \mathcal{C}$. □

Proposition 4.13. *Let $f : X \to Y$ be a group homomorphism. Then we have*

$$f(\mathcal{A} \circ \mathcal{B}) = f(\mathcal{A}) \circ f(\mathcal{B}) \text{ for any } \mathcal{A},\ \mathcal{B} \in \mathcal{O}(X).$$

Proof. Let $y \in Y$ such that $f^{-1}(y) \neq \varnothing$, say $y = f(x)$. Then we have
$f(\mathcal{A} \circ \mathcal{B})(y) = \bigvee_{x \in X,\ y = f(x)} \mathcal{A} \circ \mathcal{B}(x)$
$= \bigvee_{x \in X,\ y = f(x)} (\bigvee_{x_1 x_2 = x} [\mathcal{A}(x_1) \wedge \mathcal{B}(x_2)])$
$= \bigvee_{x_1,\ x_2 \in X,\ y = f(x_1) f(x_2)} [\mathcal{A}(x_1) \wedge \mathcal{B}(x_2)])$,
[Since f is a homomorphism]
$f(\mathcal{A}) \circ f(\mathcal{B})(y) = \bigvee_{y_1 y_2 = y} [f(\mathcal{A})(y_1) \wedge f(\mathcal{B})(y_2)]$
$= \bigvee_{y_1 y_2 = y} [(\bigvee_{x_1 \in X,\ y_1 = f(x_1)} \mathcal{A}(x_1)) \wedge (\bigvee_{x_2 \in X,\ y_2 = f(x_2)} \mathcal{B}(x_2))]$
$= \bigvee_{x_1,\ x_2 \in X,\ y = f(x_1) f(x_2)} [\mathcal{A}(x_1) \wedge \mathcal{B}(x_2)])$.
Thus the result holds. □

The following is an immediate consequence of Result 4.3 and Definition 4.10 (ii).

Proposition 4.14. *If $\mathcal{G}$ is an octahedron group in a group X, then so is $\mathcal{G}^{-1}$.*

Definition 4.15 ([36]). Let X be a group and let $\mathcal{N} \in \mathcal{O}(X)$. Then $\mathcal{N}$ is called an *octahedron normal group* in X, if it is an octahedron group in X and $\mathcal{N}(xy) = \mathcal{N}(yx)$ for any $x,\ y \in X$.

Result 4.16 (Propositions 22 and 23, [36]). *Let X be a group and let $\mathcal{N}$ be an octahedron normal group in X. If $\mathcal{A}$ is an octahedron group in X, then $\mathcal{A}\circ\mathcal{N} = \mathcal{N}\circ\mathcal{A}$ is an octahedron group in X.*

5. Octahedron topological groups

In this section, we define an octahedron topological group in a group X in Forster's sense and find its characterization (See Theorem 5.7), and we give the sufficient conditions which the inverse image and the image of an octahedron set under a group homomorphism is an octahedron topological group. Also, we introduce the concept of a relative octahedron homeomorphism and obtain some of its properties.

Proposition 5.1. *Let $\mathcal{G}$ be an octahedron group in a group X. Let $\alpha : X \times X \to X$ and $\beta : X \to X$ be the mappings respectively defined as follows:*

$$\alpha(x,y) = xy \text{ for each } (x,y) \in X \times X \text{ and } \beta(x) = x^{-1} \text{ for each } x \in X.$$

Then $\alpha(\mathcal{G}\times\mathcal{G}) \subset \mathcal{G}$ and $\beta(\mathcal{G}) \subset \mathcal{G}$.

Proof. Let $z \in X$. Then we get

$$\begin{aligned}\alpha(\mathcal{G}\times\mathcal{G})(z) &= \bigvee_{(x,y)\in\alpha^{-1}(z)}[\mathcal{G}(x)\wedge\mathcal{G}(y)]\\ &\leq \bigvee_{(x,y)\in\alpha^{-1}(z)}\mathcal{G}(xy) \text{ [By Definition 4.1 (i)]}\\ &= \mathcal{G}(z),\end{aligned}$$

$$\begin{aligned}\beta(\mathcal{G})(z) &= \bigvee_{x\in\beta^{-1}(z)}\mathcal{G}(x)\\ &= \bigvee_{x\in\beta^{-1}(z)}\mathcal{G}(x^{-1}) \text{ [By Result 4.2]}\\ &= \mathcal{G}(z). \text{ [Since } z=\beta(x)=x^{-1}]\end{aligned}$$

Thus $\alpha(\mathcal{G}\times\mathcal{G}) \subset \mathcal{G}$ and $\beta(\mathcal{G}) \subset \mathcal{G}$. □

Remark 5.2. (1) From Proposition 5.1, it is obvious that if (X,τ) is an octahedron topological space, then $(\mathcal{G},\tau_{\mathcal{G}})$ is a octahedron subspace of (X,τ) and $(\mathcal{G}\times\mathcal{G},\tau_{\mathcal{G}}\times\tau_{\mathcal{G}})$ is an octahedron subspace of the octahedron product space $(X\times X,\tau\times\tau)$. Furthermore, from Remark 3.3, we can easily see that $(\widetilde{G},\tau_{IV_{\widetilde{G}}})$ is an interval-valued fuzzy subspace of (X,τ_{IV}), $(\bar{G},\tau_{IF_{\bar{G}}})$ is an intuitionistic fuzzy subspace of (X,τ_{IF}), (G,τ_{F_G}) is an fuzzy subspace of (X,τ_F) and $(\widetilde{G}\times\widetilde{G},\tau_{IV_{\widetilde{G}}}\times\tau_{IV_{\widetilde{G}}})$ is an interval-valued fuzzy subspace of $(X\times X,\tau_{IV}\times\tau_{IV})$, $(\bar{G}\times\bar{G},\tau_{IF_{\bar{G}}}\times\tau_{IF_{\bar{G}}})$ is an intuitionistic fuzzy subspace of $(X\times X,\tau_{IF}\times\tau_{IF})$, $(G\times G,\tau_{F_G}\times\tau_{F_G})$ is an fuzzy subspace of $(X\times X,\tau_F\times\tau_F)$.

(2) Let $\mathcal{G}$ be an octahedron group in a group X. Then from Proposition 4.14, $\mathcal{G}^{-1}$ is an octahedron group in X. Moreover, we can easily check that

$$\alpha(\mathcal{G}^{-1}\times\mathcal{G}^{-1}) \subset \mathcal{G}^{-1} \text{ and } \beta(\mathcal{G}^{-1}) \subset \mathcal{G}^{-1}.$$

(3) Let X be a group, let $\mathcal{N}$ be an octahedron normal group in X and let $\mathcal{A}$ be an octahedron group in X. Then from Result 4.16, $\mathcal{A}\circ\mathcal{N}$ is an octahedron group in X. Moreover, we can easily see that

$$\alpha(\mathcal{A}\circ\mathcal{N}\times\mathcal{A}\circ\mathcal{N})\subset\mathcal{A}\circ\mathcal{N} \text{ and } \beta(\mathcal{A}\circ\mathcal{N})\subset\mathcal{A}\circ\mathcal{N}.$$

Definition 5.3 ([3, 27]). (1) Let X be a group and let $\tau\in FT_L(X)$. Let G be a fuzzy group in X and let (G,τ_G) be a fuzzy subspace of (X,τ). Then G is called a *fuzzy topological group*, if it satisfies the following axioms:

(i) $\alpha:(G\times G,\tau_G\times\tau_G)\to(G,\tau_G)$ is relatively fuzzy continuous,

(ii) $\beta:(G,\tau_G)\to(G,\tau_G)$ is relatively fuzzy continuous.

(2) Let X be a group and let $\tau\in IFT_L(X)$. Let $\bar{G}$ be an intuitionistic fuzzy group in X and let $(\bar{G},\tau_{\bar{G}})$ be an intuitionistic fuzzy subspace of (X,τ). Then $\bar{G}$ is called an *intuitionistic fuzzy topological group*, if it satisfies the following axioms:

(i) $\alpha:(\bar{G}\times\bar{G},\tau_{\bar{G}}\times\tau_{\bar{G}})\to(\bar{G},\tau_{\bar{G}})$ is relatively intuitionistic fuzzy continuous,

(ii) $\beta:(\bar{G},\tau_{\bar{G}})\to(\bar{G},\tau_{\bar{G}})$ is relatively intuitionistic fuzzy continuous.

Definition 5.4 ([3, 27]). Let X be a group and let $\tau\in IVT_L(X)$. Let $\widetilde{G}$ be an interval-valued fuzzy group in X and let $(\widetilde{G},\tau_{\widetilde{G}})$ be an interval-valued fuzzy subspace of (X,τ). Then $\widetilde{G}$ is called an *interval-valued topological group*, if it satisfies the following axioms:

(i) $\alpha:(\widetilde{G}\times\widetilde{G},\tau_{\widetilde{G}}\times\tau_{\widetilde{G}})\to(\widetilde{G},\tau_{\widetilde{G}})$ is relatively interval-valued fuzzy continuous,

(ii) $\beta:(\widetilde{G},\tau_{\widetilde{G}})\to(\widetilde{G},\tau_{\widetilde{G}})$ is relatively interval-valued fuzzy continuous.

Definition 5.5. Let X be a group and let $\tau\in OT_L(X)$. Let $\mathcal{G}$ be an octahedron group in X and let $(\mathcal{G},\tau_{\mathcal{G}})$ be an octahedron subspace of (X,τ). Then $\mathcal{G}$ is called an *octahedron topological group*, if it satisfies the following axioms:

(i) $\alpha:(\mathcal{G}\times\mathcal{G},\tau_{\mathcal{G}}\times\tau_{\mathcal{G}})\to(\mathcal{G},\tau_{\mathcal{G}})$ is relatively octahedron continuous,

(ii) $\beta:(\mathcal{G},\tau_{\mathcal{G}})\to(\mathcal{G},\tau_{\mathcal{G}})$ is relatively octahedron continuous.

An octahedron group structure and a relative octahedron topology are said to be *compatible*, if they satisfy axioms (i) and (ii).

Remark 5.6. From Remarks 3.3, 5.2 and Definitions 5.3, 5.4, 5.5, we can easily check that $\mathcal{G}$ is an octahedron topological group, then $\widetilde{G}$ is an interval-valued fuzzy topological group, $\bar{G}$ is an intuitionistic fuzzy topological group and G is a fuzzy topological group.

Theorem 5.7. *Let X be a group with $\tau\in OT_L(X)$, let $\mathcal{G}\in\mathcal{O}(X)$ and let $\delta: X\times X\to X$ be the mapping defined by: for each $(x,y)\in X\times X$,*

$$\delta(x,y)=xy^{-1}.$$

Then $\mathcal{G}$ is octahedron topological group in X if and only if the mapping $\delta:(\mathcal{G}\times\mathcal{G},\tau_{\mathcal{G}}\times\tau_{\mathcal{G}})\to(\mathcal{G},\tau_{\mathcal{G}})$ is relatively octahedron continuous.

Proof. Suppose $\mathcal{G}$ is octahedron topological group in X. It is clear that the identity mapping $id:(\mathcal{G},\tau_{\mathcal{G}})\to(\mathcal{G},\tau_{\mathcal{G}})$ and the mapping $\beta:(\mathcal{G},\tau_{\mathcal{G}})\to(\mathcal{G},\tau_{\mathcal{G}})$ are relatively octahedron continuous. Let $f=id\times\beta$. Then by Corollary 3.28, $f:(\mathcal{G}\times\mathcal{G},\tau_{\mathcal{G}}\times\tau_{\mathcal{G}})\to(\mathcal{G}\times\mathcal{G},\tau_{\mathcal{G}}\times\tau_{\mathcal{G}})$ is relatively octahedron continuous. On the other hand, it is obvious that $\delta=\alpha\circ f$. Thus δ is relatively octahedron continuous.

Conversely, suppose δ is relatively octahedron continuous. Consider the canonical injection $i : X \to X \times X$ defined by: for each $y \in X$,

$$i(y) = (e, y).$$

Then by Definition 2.6 (ii) and Result 4.2, $i(\mathcal{G}) \subset \mathcal{G} \times \mathcal{G}$. Since $i : (X, \tau) \to (X \times X, \tau \times \tau)$ is octahedron continuous, the mapping $i : (\mathcal{G}, \tau_{\mathcal{G}}) \to (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}})$ is relatively octahedron continuous. Moreover, $\beta = \delta \circ i$. Thus β is relatively octahedron continuous. So $\beta \circ \delta$ is relatively octahedron continuous and $\alpha = \beta \circ \delta$. Hence α is relatively octahedron continuous. Therefore $\mathcal{G}$ is an octahedron topological group. □

We obtain the followings.

Corollary 5.8. *Let $\mathcal{G}$ be an octahedron group in a group X. Then $\mathcal{G}^{-1}$ is an octahedron topological group in X.*

Proof. The proof is similar to Theorem 5.7. □

Corollary 5.9. *Let X be a group, let $\mathcal{N}$ be an octahedron normal group in X and let $\mathcal{A}$ be an octahedron group in X. Then $\mathcal{A} \circ \mathcal{N}$ is an octahedron topological group in X.*

Proof. The proof is similar to Theorem 5.7. □

Definition 5.10. Let (X, τ), (Y, γ) be two octahedron topological spaces and let let $f : X \to Y$ be a bijective mapping. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be an octahedron subspace of (X, τ) and let $(\mathcal{B}, \gamma_{\mathcal{B}})$ be an octahedron subspace of (Y, γ).

(i) $f : (X, \tau) \to (Y, \gamma)$ is called an *octahedron homeomorphism*, if it is octahedron continuous and open,

(ii) $f : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \gamma_{\mathcal{B}})$ is called a *relative octahedron homeomorphism*, if it is relatively octahedron continuous and open.

Remark 5.11. Let $\mathcal{G}$ be an octahedron topological group in a group X with $\tau \in OT_L(X)$ and for a fixed $a \in X$, let r_a and l_a be translations. Then $r_a : (\mathcal{G}, \tau_{\mathcal{G}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ and $l_a : (\mathcal{G}, \tau_{\mathcal{G}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ are not relatively octahedron continuous, in general. However, we have the special case.

Proposition 5.12. *Let X be be a group with $\tau \in OT_L(X)$ and let $\mathcal{G}$ be an octahedron topological group. Then the translations r_a, $l_a : (\mathcal{G}, \tau_{\mathcal{G}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ are relative octahedron homeomorphisms for each $a \in \mathcal{G}_e$.*

Proof. It is clear that la is bijective. Let $a \in \mathcal{G}_e$. Then clearly, by Proposition 4.9, $l_a(\mathcal{G}) = \mathcal{G}$. Furthermore, $l_a = \alpha \circ i$, where $i : X \to X \times X$ is the injection defined by $i(y) = (a, y)$ for each $y \in X$. Since $a \in \mathcal{G}_e$, $\mathcal{G}(a) = \mathcal{G}(e)$. By Result 4.2, $\mathcal{G}(a) \geq \mathcal{G}(x)$ for each $x \in X$. Thus by Proposition 3.34, $i : (\mathcal{G}, \tau_{\mathcal{G}}) \to (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}})$ is relative octahedron continuous. So by the hypothesis, α is relative octahedron continuous. Hence l_a is relative octahedron continuous. Since $l_a^{-1} = l_{a^{-1}}$, l_a is relatively octahedron open. Therefore l_a is a relative octahedron homeomorphism.

The proof of a relative octahedron homeomorphism of r_a is similar. □

Suppose $f : X \to Y$ is a group homomorphism with $\gamma \in OT_L(Y)$ and $\mathcal{G}$ is an octahedron topological group in Y. Then by Definition 3.21, $f^{-1}(\gamma) \in OT_L(X)$, say $f^{-1}(\gamma) = \tau$. Since $f^{-1}(\mathcal{G}) \in OG(X)$ by Result 4.5 (2), $(f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})})$ is an octahedron subspace of an octahedron topological space (X, τ).

The following shows that the relative octahedron topology on $f^{-1}(\mathcal{G})$ and the octahedron group structure are compatible.

Proposition 5.13. *Let $f : X \to Y$ be a group homomorphism with $\gamma \in OT_L(Y)$ and $\mathcal{G}$ be an octahedron topological group in Y. Then $f^{-1}(\mathcal{G})$ is an octahedron topological group in X.*

Proof. From Definition 3.21 and Result 4.5 (2), we can easily see that (X, τ) is an octahedron topological space and $f^{-1}(\mathcal{G}) \in OG(X)$, where $\tau = f^{-1}(\gamma)$. Then $(f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})})$ is an octahedron subspace of (X, τ). By Theorem 5.7, it is sufficient to prove that the mapping $\delta_X : (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})} \times \tau_{f^{-1}(\mathcal{G})}) \to (f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})})$ is relatively octahedron continuous, where $\delta_X : X \times X \to X$ is the mapping defined by $\delta_X(x, y) = xy^{-1}$ for each $(x, y) \in X \times X$.

Let $\mathcal{U}' \in \tau_{f^{-1}(\mathcal{G})}$ and let $(x, y) \in X \times X$. By Definition 3.21 and Result 2.7 (4), $f : (X, \tau) \to (Y, \gamma)$ is octahedron continuous and $f(f^{-1}(\mathcal{G})) \subset \mathcal{G}$. Then by Proposition 3.14, $f : (f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})}) \to (\mathcal{G}, \gamma_{\mathcal{G}})$ is relatively octahedron continuous. Thus there is $\mathcal{V}' \in \gamma_{\mathcal{G}}$ such that $f^{-1}(\mathcal{V}') = \mathcal{U}'$. On the other hand, we have

$$\begin{aligned}
\delta_X^{-1}(\mathcal{U}')(x, y) &= \mathcal{U}'(\delta_X(x, y)) \text{ [By Definition 2.6 (i)]}\\
&= \mathcal{U}'(xy^{-1}) \text{ [By the definition of } \delta_X]\\
&= f^{-1}(\mathcal{V}')(xy^{-1})\\
&= \mathcal{V}'(f(xy^{-1}))\\
&= \mathcal{V}'(f(x)(f(y)^{-1}). \text{ [Since } f \text{ is a homomorphism]}
\end{aligned}$$

Since $\mathcal{G}$ is an octahedron topological group in Y, the mapping $\delta_Y : (\mathcal{G} \times \mathcal{G}, \gamma_{\mathcal{G}} \times \gamma_{\mathcal{G}}) \to (\mathcal{G}, \gamma_{\mathcal{G}})$ is relatively octahedron continuous, where $\delta_Y : Y \times Y \to Y$ is the mapping defined by $\delta_Y(y_1 y_2) = y_1 y_2^{-1}$ for each $(y_1, y_2) \in Y \times Y$. So by Corollary 3.30, the product mapping $f \times f : (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G}), \tau_{f^{-1}(\mathcal{G})} \times \tau_{f^{-1}(\mathcal{G})}) \to (\mathcal{G}, \gamma_{\mathcal{G}})$ is relatively octahedron continuous. But we get

$$\mathcal{V}'(f(xy^{-1})) = \delta_Y^{-1}(\mathcal{V}')(f(x), f(y)) = (f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')](x, y).$$

Furthermore, we have

$$\delta_X^{-1}(\mathcal{U}') \cap (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})) = (f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})).$$

Since $(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})) \in \tau_{f^{-1}(\mathcal{G})} \times \tau_{f^{-1}(\mathcal{G})}$, it is clear that

$$\delta_X^{-1}(\mathcal{U}') \cap (f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})) \in \tau_{f^{-1}(\mathcal{G})} \times \tau_{f^{-1}(\mathcal{G})}.$$

Hence δ_X is relatively octahedron continuous. Therefore $f^{-1}(\mathcal{G})$ is an octahedron topological group in X. □

The following shows that for some homomorphic images a similar situation holds.

Proposition 5.14. *Let $f : X \to Y$ be a group homomorphism with $\tau \in OT_L(X)$ and let $\mathcal{G}$ be an octahedron topological group in X. If $\mathcal{G}$ is f-invariant, then $f(\mathcal{G})$ is an octahedron topological group in Y.*

Proof. Suppose $\mathcal{G}$ is f-invariant. Then clearly, by Proposition 4.7, $f(\mathcal{G}) \in OG(Y)$. From Definition 3.23, it is obvious that $f(\tau) \in OT_L(Y)$, say $f(\tau) = \gamma$. It is sufficient to prove that the mapping $\delta_Y : (f(\mathcal{G}) \times f(\mathcal{G}), \gamma_{f(\mathcal{G})} \times \gamma_{f(\mathcal{G})}) \to (f(\mathcal{G}), \gamma_{f(\mathcal{G})})$ is relatively octahedron continuous.

From Definitions 3.5 (ii) and 3.23, it is clear that $f : (X, \tau) \to (Y, \gamma)$ is octahedron open. Let $\mathcal{U}' \in \tau_{\mathcal{G}}$. Then there is $\mathcal{U} \in \tau$ such that $\mathcal{U}' = \mathcal{U} \cap \mathcal{G}$. Since $\mathcal{G}$ is f-invariant, $f(\mathcal{U}') = f(\mathcal{U}) \cap f(\mathcal{G})$. Since f is octahedron open, $f(\mathcal{U}) \in \gamma$. Thus $f(\mathcal{U}') \in \gamma_{f(\mathcal{G})}$. So $f : (\mathcal{G}, \tau_{\mathcal{G}} \to (f(\mathcal{G}), \gamma_{f(\mathcal{G})})$ is relatively octahedron open. By Proposition 3.32, the product mapping $f \times f : (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}) \to (f(\mathcal{G}) \times f(\mathcal{G}), \gamma_{f(\mathcal{G})} \times \gamma_{f(\mathcal{G})})$ is relatively octahedron open.

Now let $\mathcal{V}' \in \gamma_{f(\mathcal{G})}$ and let $(x, y) \in X \times X$. Then we have

$$(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')](x, y) = \mathcal{V}'(f(x)(f(y))^{-1}) = (\delta_X^{-1} \circ f^{-1})(\mathcal{V}')(x, y),$$

where $\delta_X : X \times X \to X$ is the mapping defined by $\delta_X(x, y) = xy^{-1}$ for each $(x, y) \in X \times X$. Since $\mathcal{G}$ is an octahedron topological group in X, $\delta_X : (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ is relatively octahedron continuous and $f : (\mathcal{G}, \tau_{\mathcal{G}}) \to (f(\mathcal{G}), \gamma_{(\mathcal{G})})$ is relatively octahedron continuous. Since $\mathcal{G}$ is f-invariant, we get

$$(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f(\mathcal{G}) \times f(\mathcal{G})) = (f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (\mathcal{G} \times \mathcal{G}).$$

Since $(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (\mathcal{G} \times \mathcal{G}) \in \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}$, it is clear that

$$(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f(\mathcal{G}) \times f(\mathcal{G})) \in \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}.$$

Since $f \times f$ is relatively octahedron open,

$$(f \times f)(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f(\mathcal{G}) \times f(\mathcal{G})) \in \gamma_{f(\mathcal{G})} \times \gamma_{f(\mathcal{G})}.$$

On the other hand, we have

$$(f \times f)(f \times f)^{-1}[\delta_Y^{-1}(\mathcal{V}')] \cap (f(\mathcal{G}) \times f(\mathcal{G})) = \delta_Y^{-1}(\mathcal{V}') \cap (f(\mathcal{G}) \times f(\mathcal{G})).$$

Thus $\delta_Y^{-1}(\mathcal{V}') \cap (f(\mathcal{G}) \times f(\mathcal{G})) \in \gamma_{f(\mathcal{G})} \times \gamma_{f(\mathcal{G})}$. So δ_Y is relatively octahedron continuous. Hence $f(\mathcal{G})$ is an octahedron topological group in Y. □

Remark 5.15. Let X be a group with $\tau \in OT_L(X)$, let $\mathcal{G}$ be an octahedron topological group, let N be a normal subgroup of X and let φ be the canonical mapping of X onto the quotient group X/N. Then we can easily check that if $\mathcal{G}$ is constant on N, then $\mathcal{G}$ is φ-invariant and thus by Proposition 4.7, $\varphi(\mathcal{G}) \in OG(X/N)$.

In this case, $\varphi(\mathcal{G})$ is called an *octahedron quotient group* in X/N and denoted by $\mathcal{G}/N$.

Proposition 5.16. *Let X be a group with $\tau \in OT_L(X)$, let $\mathcal{G}$ be an octahedron topological group, let N be a normal subgroup of X and let φ be the canonical mapping of X onto the quotient group X/N. Let $\gamma = \varphi(\tau) \in OT_L(X/N)$. If $\mathcal{G}$ is constant on N, then the quotient group $\mathcal{G}/N$ is an octahedron topological group in X/N.*

In this case, $\varphi(\tau)$ is called the *octahedron quotient topology* on X/N and G/N is called an *octahedron quotient topological group* in X/N.

Proof. The proof is similar to one of Proposition 5.14. □

Proposition 5.17. *Let $f : X \to Y$ be a group epimorphism with $\tau \in OT_L(X)$ and $\gamma \in OT_L(Y)$. Let $f : (X, \tau) \to (Y, \gamma)$ be octahedron continuous and open. Let $\mathcal{G}$ be an octahedron topological group in X such that $\mathcal{G}$ is constant on the kernel $f^{-1}(e)$ of f and let the octahedron quotient group $\mathcal{G}/f^{-1}(e)$ have the octahedron quotient topology. Then*

(1) *the octahedron groups $\mathcal{G}/f^{-1}(e)$ and $f(\mathcal{G})$ are octahedron topological groups in $X/f^{-1}(e)$ and Y respectively,*

(2) *the canonical isomorphism $\bar{f}$ of $X/f^{-1}(e)$ onto Y is a relative homeomorphism of $\mathcal{G}/f^{-1}(e)$ onto $f(\mathcal{G})$.*

Proof. (1) It is clear that $f^{-1}(e)$ is a normal subgroup of X. Then by Proposition 5.16, $\mathcal{G}/f^{-1}(e)$ is an octahedron topological group in $X/f^{-1}(e)$. Let $\mathcal{V} \in f(\tau)$. Then $f^{-1}(\mathcal{V}) \in \tau$. Since f is sujective, $f(f^{-1}(\mathcal{V})) = \mathcal{V}$ by Result 2.7 (4). Thus $\mathcal{V} \in \gamma$. So $f(\tau) \subset \gamma$. Now let $\mathcal{V} \in \gamma$. Since f is octahedron continuous, $f^{-1}(\mathcal{V}) \in \tau$. Then $\mathcal{V} \in f(\tau)$. Thus $\gamma \subset f(\tau)$. So $f(\tau) = \gamma$. Moreover, we can easily see that $\mathcal{G}$ is f-invariant. Hence by Proposition 5.14, $f(\mathcal{G})$ is an octahedron topological group in Y.

(2) Let $\mathcal{V}^{'} \in \gamma_{f(\mathcal{G})}$ and let φ be the canonical homomorphismn of X onto $X/f^{-1}(e)$. Then $f^{-1}(\mathcal{V}^{'}) = \varphi^{-1}(\bar{f}^{-1}(\mathcal{V}^{'}))$. Since f is relatively octahedron continuous, $f^{-1}(\mathcal{V}^{'}) \in \tau_{\mathcal{G}}$. Thus $\varphi^{-1}(\bar{f}^{-1}(\mathcal{V}^{'})) \in \tau_{\mathcal{G}}$. So $\bar{f}^{-1}(\mathcal{V}^{'}) \in \tau_{\mathcal{G}/f^{-1}(e)}$. Hence the mapping $\bar{f} : (\mathcal{G}/f^{-1}(e)) \to (f(\mathcal{G}), \gamma_{f(\mathcal{G})})$ is relatively octahedron continuous.

Now let $\mathcal{U} \in \tau_{\mathcal{G}/f^{-1}(e)}$. Then $\varphi^{-1}(\mathcal{U}) = f^{-1}(\bar{f}(\mathcal{U}))$. Since φ is relatively octahedron continuous, $\varphi^{-1}(\mathcal{U}) \in \tau_{\mathcal{G}}$. Thus $f^{-1}(\bar{f}(\mathcal{U})) \in \tau_{\mathcal{G}}$. Since f is relatively octahedron open and surjective, $\bar{f}(\mathcal{U}) = f(f^{-1}(\bar{f}(\mathcal{U}))) \in \tau_{\mathcal{G}}$. So $\bar{f}$ is relatively octahedron open. Hence $\bar{f}$ is a relative octahedron homeomorphism. □

Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups and let X be the product group of $\{X_j\}$. For each $j = 1, 2, \cdots, n$, let $\tau_j \in OT_L(X_j)$ and let $\mathcal{G}_j$ be an octahedron topological group. Let $\mathcal{G} = \Pi_{j=1}^{n} \mathcal{G}_j$ be the octahedron product set in X defined by: for each $x = (x_1, x_2, \cdots, x_n) \in X$,

$$\mathcal{G}(x) = \mathcal{G}_1(x_1) \wedge \mathcal{G}_2(x_2) \wedge \cdots \wedge \mathcal{G}_n(x_n).$$

Proposition 5.18. *Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups and let X be the product group of $\{X_j\}$. For each $j = 1, 2, \cdots, n$, let $\mathcal{G}_j$ be an octahedron group in X_j. Then $\mathcal{G} = \Pi_{j=1}^{n} \mathcal{G}_j$ is an octahedron group in X.*

In this case, $\mathcal{G}$ is called an *octahedron product group* of $\{\mathcal{G}_j\}$.

Proof. The proof is easy from Result 4.3 and the definition of octahedron product set. □

The following shows that the relative octahedron topology on $\mathcal{G}$ and the octahedron group structure are compatible.

Proposition 5.19. *Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups with $\tau_j \in OT_L(X_j)$ and let X be the product group of $\{X_j\}$. For each $j = 1, 2, \cdots, n$, let $\mathcal{G}_j$ be an octahedron group in X_j and let (X, τ) be the octahedron topological space, where τ is the octahedron product topology of $\{\tau_j\}$. Then the octahedron product group $\mathcal{G}$ is an octahedron topological group in X.*

In this case, $\mathcal{G}$ is called an *octahedron product topological group* of $\{\mathcal{G}_j\}$.

Proof. Let $\delta_1 : X \times X \to \Pi_{j=1}^n (X_j \times X_j)$ be two mapping defined as follows: for each $(x, y) = ((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) \in X \times X$,

$$\delta_1(x, y) = ((x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)).$$

Then by Theorem 3.27, $\delta_1 : (X \times X, \tau \times \tau) \to (\Pi_{j=1}^n (X_j \times X_j), \Pi_{j=1}^n (\tau_j \times \tau_j))$ is octahedron continuous. It is obvious that $\delta_1(\mathcal{G} \times \mathcal{G}) \subset \Pi_{j=1}^n (\mathcal{G}_j \times \mathcal{G}_j)$. Thus by Proposition 3.14, $\delta_1 : (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}) \to (\Pi_{j=1}^n (\mathcal{G}_j \times \mathcal{G}_j), \Pi_{j=1}^n ((\tau_j)_{\mathcal{G}_j} \times (\tau_j)_{\mathcal{G}_j}))$ is relatively octahedron continuous. Let $\delta_2 : \Pi_{j=1}^n (X_j \times X_j) \to X$ be mapping defined as follows: for each $(x, y) = ((x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)) \in \Pi_{j=1}^n (X_j \times X_j)$,

$$\delta_2(x, y) = (x_1 y_1^{-1}, x_2 y_2^{-1}, \cdots, x_n y_n^{-1}).$$

Then we can easily check that $\delta_2 : (\Pi_{j=1}^n (X_j \times X_j), \Pi_{j=1}^n (\tau_j \times \tau_j) \to (X, \tau)$ is octahedron continuous and $\delta_2(\Pi_{j=1}^n (\mathcal{G}_j \times \mathcal{G}_j)) \subset \mathcal{G}$. Thus by Corollary 3.30, $\delta_2 : (\Pi_{j=1}^n (\mathcal{G}_j \times \mathcal{G}_j), \Pi_{j=1}^n ((\tau_j)_{\mathcal{G}_j} \times (\tau_j)_{\mathcal{G}_j}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ is relatively octahedron continuous. Now let $\delta = \delta_2 \circ \delta_1 : X \times X \to X$. Then clearly, $\delta : X \times X \to X$ be the mapping given by $\delta(x, y) = xy^{-1}$ for each $(x, y) \in X \times X$. Thus $\delta : (X \times X, \tau \times \tau) \to (X, \tau)$ is octahedron continuous and $\delta(\mathcal{G} \times \mathcal{G}) \subset \mathcal{G}$. So $\delta : (\mathcal{G} \times \mathcal{G}, \tau_{\mathcal{G}} \times \tau_{\mathcal{G}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ is relatively octahedron continuous. Hence $\mathcal{G}$ is an octahedron topological group in X. □

The following may be considered as the consequence combined to Propositions 5.16 and 5.18.

Proposition 5.20. *Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups with $\tau_j \in OT_L(X_j)$. For each $j = 1, 2, \cdots, n$, let N_j be be a normal subgroup of X_j and let $\mathcal{G}_j$ be an octahedron topological group in X_j such that $\mathcal{G}_j$ is constant on N_j. Let $X = \Pi_{j=1}^n X_j$ with $\tau = \Pi_{j=1}^n \tau_j \in OT_L(X)$, let $N = \Pi_{j=1}^n N_j$ and let X/N be the quotient group such that ζ is the octahedron quotient topology on X/N. For each $j = 1, 2, \cdots, n$, let η_j be the octahedron quotient topology on X_j/N_j. Suppose $\mathcal{G} = \Pi_{j=1}^n \mathcal{G}_j$ is an octahedron topological group in X. Then the canonical isomorphism i of X/N onto $\Pi_{j=1}^n (X_j/N_j)$ is a relative octahedron homeomorphism of octahedron quotient topological group $\mathcal{G}/N$ onto the octahedron product topological group $\Pi_{j=1}^n (\mathcal{G}_j/N_j)$.*

Proof. Let $\varphi : X \to X/N$ be the canonical epimorphism defined by $\varphi(x) = [x]$ for each $x \in X$ and for each $j = 1, 2, \cdots, n$, let $\varphi_j : X_j \to X_j/N_j$ be the canonical epimorphism defined by $\varphi(x_j) = [x_j]$ for each $x_j \in X_j$. Let $\Pi_{j=1}^n \varphi_j : X \to \Pi_{j=1}^n (X_j/N_j)$ be the product surjective mapping given by:

$$\Pi_{j=1}^n \varphi_j(x) = \Pi_{j=1}^n [x_j] \text{ for each } x = (x_1, x_2, \cdots, x_n) \in X.$$

Then clearly, $\Pi_{j=1}^n \varphi_j = i \circ \varphi$. Let $[x] \in X/N$. Then we have

$$\begin{aligned}\mathcal{G}/N([x]) &= \mathcal{G}(x) = \Pi_{j=1}^n \mathcal{G}_j(x_1, x_2, \cdots, x_n) \\ &= \bigwedge_{j=1}^n \mathcal{G}_j(x_j) \\ &= \bigwedge_{j=1}^n \mathcal{G}_j/N_j([x_j])\end{aligned}$$

$= \Pi_{j=1}^{n}\mathcal{G}_j/N_j(i([x]))$.
On the other hand, by Propositions 5.16 and 5.18, $\mathcal{G}/N$ and $\Pi_{j=1}^{n}(\mathcal{G}_j/N_j)$ are octahedron topological groups. Let $\eta = \Pi_{j=1}^{n}\eta_j$ be the relative octahedron topology on $\Pi_{j=1}^{n}X_j/N_j$ and let $\mathcal{V}^{'} \in \eta$. Then we get

$$(i \circ \varphi)^{-1}(\mathcal{V}^{'}) = \varphi^{-1}(i^{-1}(\mathcal{V}^{'})) = (\Pi_{j=1}^{n}\varphi_j)^{-1}(\mathcal{V}^{'}).$$

By Propositions 3.27 and 3.14, $\Pi_{j=1}^{n}\varphi_j$ is relatively octahedron continuous. Thus $\varphi^{-1}(i^{-1}(\mathcal{V}^{'}) = (\Pi_{j=1}^{n}\varphi_j)^{-1}(\mathcal{V}^{'}) \in \tau_{\mathcal{G}}$. So $(i \circ \varphi)^{-1}(\mathcal{V}^{'}) \in \tau_{\mathcal{G}}$. Since φ is relatively octahedron open and surjective, $i^{-1}(\mathcal{V}^{'}) \in \zeta_{\mathcal{G}/N}$. Hence i is relatively octahedron continuous.

Now let $\mathcal{U}^{'} \in \zeta_{\mathcal{G}/N}$. Then $\varphi^{-1}(\mathcal{U}^{'}) \in \tau_{\mathcal{G}}$. On the other hand, we get

$$(\Pi_{j=1}^{n}\varphi_j)(\varphi^{-1}(\mathcal{U}^{'})) = i(\mathcal{U}^{'}).$$

Since $\Pi_{j=1}^{n}\varphi_j$ is the product of relatively octahedron open mappings, $\Pi_{j=1}^{n}\varphi_j$ is relatively octahedron open by Proposition 3.32. Thus $(\Pi_{j=1}^{n}\varphi_j)(\varphi^{-1}(\mathcal{U}^{'})) \in \eta_{\Pi_{j=1}^{n}(\mathcal{G}_j/N_j)}$. So $i(\mathcal{U}^{'}) \in \eta_{\Pi_{j=1}^{n}(\mathcal{G}_j/N_j)}$. Hence i is relatively octahedron open. Therefore i is a relative octahedron homeomorphism. □

6. Conclusions

First of all, we obtained further properties in an octahedron topological space and some properties in an octahedron group which are necessary to discuss with properties of an octahedron topological group. Next, we defined an octahedron topological group in the sense of Forster, and obtained its characterization and some of its properties. In particular, we found the sufficient conditions which the preimage and the image of an octahedron set under an group homomorphism are an octahedron topological group. Also, we introduced the concept of a relative octahedron homeomorphism and investigated some of its properties.

In the future, we hope that one can the notion of octahedron sets apply to a semigroup, a BCK/BCI-algebra, an octahedron ideal topological group and decision-making problems. Moreover, we expect that one study interval-valued fuzzy topological groups of Forster's sense.

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S. H. Han (shhan235@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea

J. G. Lee (jukolee@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea

G. Şenel (g.senel@amasya.edu.tr)
Department of Mathematics, University of Amasya, Turkey

Minseok Cheong (poset@kau.ac.kr)
School of Liberal Arts and Sciences, Korea Aerospace University, Korea

K. Hur (kulhur@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea